European Journal of Science and Mathematics Education, 2026, **14**(3), 306-321
ISSN: 2301-251X (Online)

OPEN ACCESS

Research Article

Teachers' mathematical work on quadrilaterals area with digital technology

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Citation: Salazar, J. V. F., García-Cuéllar, D. J., Henríquez-Rivas, C., & Kuzniak, A. (2026). Teachers' mathematical work on quadrilaterals area with digital technology. *European Journal of Science and Mathematics Education, 14*(3), 306-321. https://doi.org/10.30935/scimath/18435

ARTICLE INFO

Received: 6 Aug 2025

Accepted: 25 Jan 2026

ABSTRACT

The aim of this paper is to analyze the mathematical work done by teachers when interacting with a GeoGebra application in a task on the area of quadrilaterals. The theoretical framework is focused on the mathematical working space (MWS) theory. The research approach is qualitative, based on a case study. The research subjects are two high school mathematics teachers. Results show that, in the personal MWS of both subjects, iconic visualization and pragmatic proof prevail, and the activation of the semiotic and discursive genesis is largely highlighted. Also, the type of device used has been proven to generate different MWS.

Keywords: mathematical work, teacher training, quadrilaterals, areas, qualitative research

INTRODUCTION

Literature shows difficulties encountered by basic education students when solving tasks related to calculating the area of figures (Castillo, 2018; Herendiné, 2016; Tan-Sisman & Aksu, 2009), such as confusing area with perimeter, focusing on learning formulas, and emphasizing numerical techniques over figure exploration, either with physical resources or with the use of digital technology (Ng & Sinclair, 2015). This paper presents the analysis of the mathematical work developed by teachers when interacting with a dynamic geometry software or dynamic geometry environments and solving a task on area of quadrilaterals.

Some research focuses on teacher training in mathematics, especially in the domain of geometry. In regards to the use of technologies in particular, there are reports on opportunities provided by the use of technology in teaching geometric topics (Gómez-Chacón et al., 2016; Henríquez-Rivas & Montoya-Delgadillo, 2016; Kuzniak & Nechache, 2021; Kuzniak et al., 2020).

Other research papers, such as those of Navarro (2002) and Vaillant (2013), determine that, among the gaps detected in teacher training, there is evidence of a lack of connection between theory and practice, little knowledge acquisition for the development of digital competencies, poor organization with curricular reforms, weak and insufficient disciplinary training, as well as a highly fragmented approach to what is taught and learned in teacher training institutions. Likewise, a need for teacher training has been reported to use technology appropriately in the classroom and design effective tasks in the geometric domain (Henríquez-Rivas et al., 2021). Therefore, a study that considers the use of technology for teaching geometry in teacher training contexts is proposed as a contribution to the discipline.

Contextual Background: Evaluations in Peru

In Peru, national evaluations are carried out to determine what basic education students learn, these are developed and implemented by the office for measuring the quality of learning (UMC in Spanish) of the Ministry of Education (MINEDU). In the evaluation carried out in 2016, participants included 490 637 students in the second year of high school (12 to 13 years old). Only 25.8% answered the question shown in **Figure 1** correctly, which is related to calculating the area measurement of quadrilaterals (trapezoid).

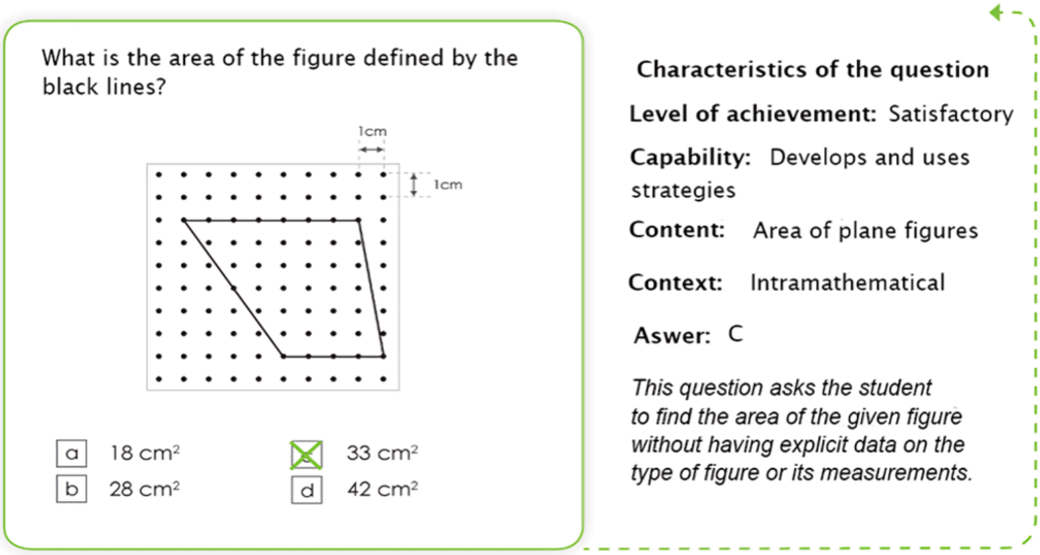


Figure 1. Question from the census evaluation of students (ECE in Spanish) 2016 (MINEDU, 2017, p. 26)

The results of the national evaluation taken by graduates from higher pedagogical education institutes–carried out in 2014–indicate that only 8.06% of the graduates reached a satisfactory level of mathematical competence (MINEDU, 2015). On the other hand, the evaluation taken by teachers who work at public schools–carried out in 2015–showed that 47.7% of a total of 19,239 teachers reached the lowest level; consequently, they did not reach the minimum teaching performance required.

Likewise, in the single national test of the 2018 teacher application contest, only 15.4% of the teachers managed to pass with minimum scores in each of the three subtests, which are reading comprehension, logical reasoning, and curricular and pedagogical knowledge of mathematics (MINEDU, 2018).

Based on the aforementioned, this situation confirms the need to contribute to mathematics teacher training so basic education students can learn better.

MATHEMATICAL WORKING SPACE

This research is based on the mathematical working space (MWS) theory, approached from the domain of geometry in particular.

The research by Kuzniak et al. (2015) presents the notions that underpin the MWS. One of these is the notion of *paradigm*, which is understood as a set of beliefs, techniques, and values shared by a scientific group. A *mathematical domain* is determined by the nature of the objects studied and the paradigms that characterize it, e.g., domain of geometry, algebra, arithmetic, analysis, etc. On the other hand, *mathematical work* is associated with the mathematical problem solving within a specific domain.

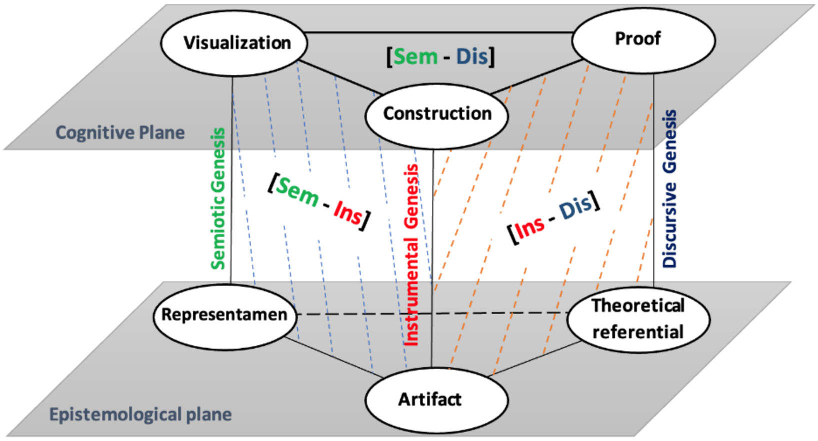


Figure 2. Genesis, horizontal, and vertical planes of the MWS (adapted from Kuzniak et al., 2016, p. 248)

Another relevant notion in the MWS theory is the *task*, which is considered as any type of mathematical problem, with explicitly and clearly established questions, which requires a predictable time for its resolution, alluding to Sierpinska (cited in Kuzniak et al., 2016). In this context, the mathematical work that a subject performs and that enables the construction of his own knowledge is a gradual, interactive and complex process, and the evolution of mathematical knowledge will depend on the proposed task and the activities carried out by the subject to solve it (Kuzniak et al., 2016).

In the MWS, the epistemological and cognitive planes are organized through three geneses. The *semiotic genesis* is the process associated with signs and representamen (or signifiers), representing the dialectical relationship between syntactic and semantic perspectives on mathematical objects, developed and organized through semiotic systems of representation. The *instrumental genesis* allows operational artifacts to be made through construction processes that contribute to achieve the mathematical work. The *discursive genesis* uses the properties of the theoretical referential system to put them at the service of mathematical reasoning. Likewise, three *vertical planes* are identified in the MWS (Kuzniak & Richard, 2014), each of which is defined by the interaction of two geneses, which activate different forms of mathematical work: semiotic and instrumental [Sem-Ins]; instrumental and discursive [Ins-Dis] and, semiotic and discursive [Sem-Dis]; this can be observed in **Figure 2**.

In the vertical plane [Sem-Ins], two forms of mathematical work are observed, one oriented towards results construction (figures and graphs), and the other towards the interpretation of the information provided by the artifacts. The vertical plane [Ins-Dis] is associated with a discursive genesis of the proof and the instrumental genesis. Finally, the vertical plane [Sem-Dis] distinguishes argumentative reasoning in coordination with visualization processes of mathematical objects involved (Kuzniak & Richard, 2014). Thus, analyzing the interactions between geneses and planes when specifying the components at stake in solving a task is called *circulation* in the MWS (Montoya-Delgadillo et al., 2014).

Three types of MWS stand out: *referential* MWS, defined by theoretical criteria with the purpose of organizing certain knowledge; *personal* MWS, defined by the way in which an individual constructs his own mathematical work; *idoine* MWS, related to a teacher or researcher who gives meaning to mathematical content designed for teaching in a given place and context (Kuzniak et al., 2022). This research highlights the personal MWS of teachers in the context of an educational activity.

In regards to the domain of geometry, three paradigms are characterized and explained below. In natural geometry–GI, objects are "material objects", lines on paper or digital lines when using digital technology such as software, etc., or even models. The usual technique in this paradigm is to design with the help of instruments such as a graduated ruler, a compass, a set square, a protractor, but also folding, cutting and tracing on paper. In the second paradigm, natural axiomatic geometry–GII, objects of study are ideal, that is, it is necessary to use definitions, axioms (proposed in Euclidean geometry). This geometry is based on GI, keeping a relationship with the sensitive space, which is why it is called "natural axiomatics". In this geometry, problems must be textual because the objects of this paradigm are textual definitions and theorems. In addition, material instruments are not used, but intellectual instruments (hypothetico-deductive reasoning) are used to construct new knowledge. Finally, in formal axiomatic geometry–GIII, objects are also ideal; the hypothetico-deductive reasoning is the origin of new knowledge. The difference between GIII and GII is that axiomatization is rigorous and formal. Based on these geometric paradigms, it is possible to understand certain difficulties in both teacher training and geometry teaching (Kuzniak & Rauscher, 2011).

Given the nature of this research, the notion of *digital artifact* should be specified, since it is used by the teachers in the study. In this sense, Flores Salazar et al. (2022) define digital artifact as a digital tool or resource used in the context of mathematical work. Digital artifacts may include software, applications, videos, among others. For example, GeoGebra is considered a digital artifact because it has a set of commands, such as propositions, that can be used by different users, such as teachers or students.

Finally, this research focuses on teacher training in the domain of geometry, particularly in the study of areas of quadrilaterals. For this purpose, the objective is to analyze the mathematical work of mathematics teachers when interacting with a GeoGebra application in a task on the area of quadrilaterals.

METHODOLOGY

Considering the aim of this research mentioned above, the research approach is qualitative, since according to Hernández et al. (2014) it provides a deep, intricate and detailed understanding of meanings, actions, phenomena, attitudes, intentions, and observable behaviors. The methodological design considers the case study based on a *unique integrated* design (Yin, 2018), where the units of analysis are the mathematical work in the domain of geometry manifested by two mathematics teachers when solving related tasks on the area of quadrilaterals using a dynamic environment of geometry.

The research subjects were two mathematics teachers who work with high school students (12 to 17 years old). As for the selection criteria for the case studies, the answers chosen were those from teachers whose mathematical work was described differently in comparison to the rest of participants in the training course. In addition, the level of detail presented in each reconfiguration and the quality of the explanation presented about their solving processes led them to be selected for the case study.

The experimental part was carried out in two synchronous virtual meetings through the Zoom platform. **Table 1** shows the activities and objectives of each meeting.

Table 1. Activities carried out in the experimental part

Number of activities	Activity	Objective
1	Introduction to GeoGebra tools	Use some GeoGebra tools in its geometric view.
2	Tasks to calculate the area of quadrilaterals	Identify the mathematical work done by teachers to calculate the area of quadrilaterals using GeoGebra

For the research data collection, different instruments were used, such as worksheets where the two teachers described their procedures to solve the task, an explanatory video made by the teachers about their procedures with the use of GeoGebra, and videos of the meetings recorded by Zoom.

For the development of this study, ethical considerations were considered in order to provide participants with a clear explanation of the nature of the study, their role, and how to use the information collected, guaranteeing data confidentiality. To this end, participants signed an informed consent protocol explaining that the data will not be used for any other purpose than the one considered in the research, and that information management does not include revealing the identity of participants or their link to the results obtained. The data analysis adopted considers elements of the proposal by Kuzniak and Nechache (2019), since it focuses on analyzing the personal MWS of participating subjects, organized according to two stages:

1. *Top-down analysis*, which describes the main actions performed when solving the task, identifying work episodes, and analyzes the circulation in the MWS.
2. *Bottom-up analysis*, which systematizes subjects' work episodes based on the MWS scheme (**Figure 2**).

Table 2. Stages of analysis of mathematics teachers' personal MWS

Stages of analysis	Analysis criteria
Top-down	Identification of work episodes Description of the work performed by the subjects in each episode Analysis of the circulation of each subject's personal MWS
Bottom-up	Synthesis and general description of the work using the MWS scheme

Table 3. Analysis criteria for MWS circulation (Henríquez-Rivas et al., 2021, p. 129)

Criterion	Component	Description
Semiotic genesis	Representation	It relates mathematical objects and their signifier elements.
	Visualization	It interprets and relates mathematical objects according to cognitive activities linked to the semiotic representation register (identification, processing, and conversions).
		The visualization process considers two levels of object visual identification (iconic visualization, non-iconic visualization).
Instrumental genesis	Artifact	It uses material, symbolic or digital artifact.
	Construction	It is based on the actions triggered by the artifacts used and the associated usage techniques.
Discursive genesis	Referential	It uses definitions, properties or theorems.
	Proof	Discursive reasoning is based on evidence (pragmatic, intellectual).

Table 2 shows a synthesis of the stages in the analysis strategy used in this research, based on Henríquez-Rivas and Kuzniak (2021, p. 1557).

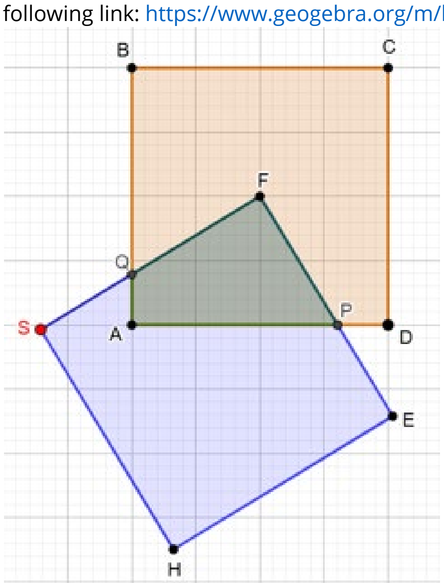
In addition, the MWS circulation criteria proposed by Henríquez-Rivas et al. (2021) were used to characterize these personal MWS, which are shown in **Table 3**.

It is worth mentioning that, to identify episodes and analyze the personal MWS circulation, triangulation is considered among the researchers who are the authors of this study, given their experience and training, as well as their different backgrounds and experience, working in universities as academics and researchers. The following is the task proposed to the teachers and the expected strategy.

Proposed Task

Table 4 shows the proposed task. It should be noted that, by clicking on the link, the application made to perform the task will appear, showing that it did not have a toolbar or an input bar, so that teachers focus on the reconfiguration by dragging point S.

Table 4. Task on the measurement of the area of quadrilaterals

Problem	Description
Go to the following link: https://www.geogebra.org/m/hhesv5tx 	Squares ABCD and EFSH are congruent, and vertex F in EFSH is at the center of ABCD and rotates around it. Drag point S so that the intersection of the squares forms the configuration of a triangle, a square and a quadrilateral. What is the relation between the measurement of the intersection area of the squares and the area of square ABCD in the different configurations? Explain your answer.

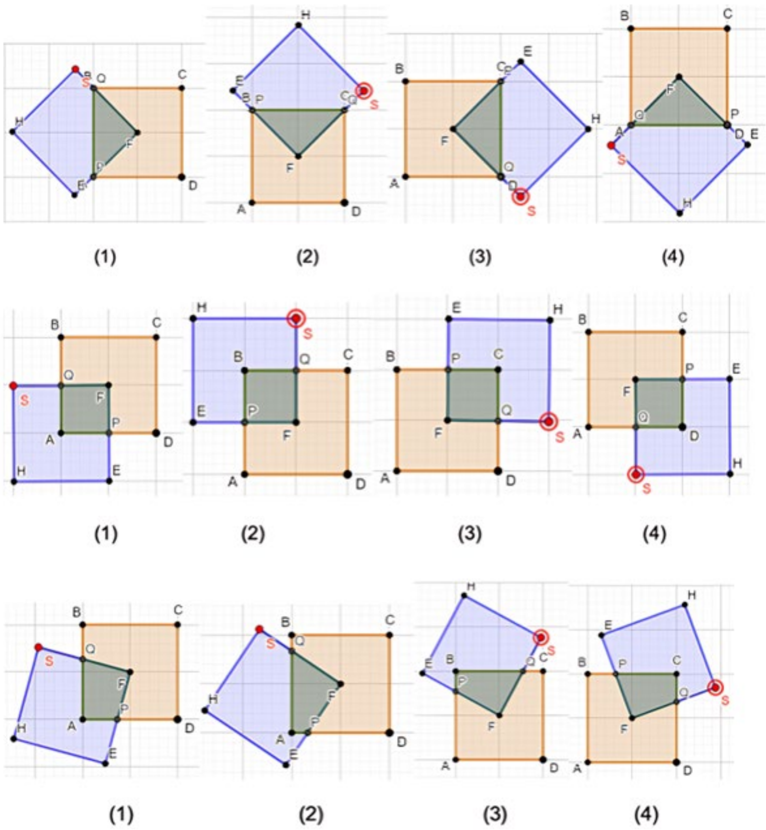


Figure 3. Configurations of the intersection of squares ABCD and EFSH (generated by the authors)

The purpose of the task is for teachers to prove that, from the reconfigurations created by moving point S, the area of intersection of squares ABCD and EFSH is always 1/4 of the area of square ABCD.

To this end, 3 configurations, called subtasks, are identified: configuration of a triangle (subtask 1), of a square (subtask 2) and of a trapezoid (subtask 3), generated by rotating point S around fixed point F, and a conclusion or validation (subtask 4). **Figure 3** shows that there are 3 possible configurations (triangle, square, and trapezoid). In UK English, a trapezoid is defined as a quadrilateral with no pair of parallel sides and therefore constitutes an irregular quadrilateral).

RESULTS

Below we present the analysis of the responses given by two teachers that we called D1 and D2. To analyze the personal MWS of D1 and D2, the stages described in **Table 2** (top-down and bottom-up analysis) and the criteria to analyze MWS circulation (see **Table 3**) were considered.

Case 1. D1's Personal Mathematical Working Space

In the top-down analysis, the following episodes (*En*) are identified in each subtask (ST) of the work done by D1: in the first subtask called configuration as a triangle, two episodes (E1 and E2) are evidenced; in the second subtask called configuration as a square, one episode (E3) is evidenced; in the third subtask called configuration as a trapezoid, two episodes (E4 and E5) are evidenced; and finally, in the fourth phase called general conclusion, one episode (E6) is evidenced.

Table 5 shows the subtasks, episodes and mathematical actions that were identified in D1's mathematical work.

In ST1, E1 and D1 moves point S by dragging it in such a way that the intersection of the areas of the squares creates a triangle (PFQ) and divides the latter into two smaller triangles. D1 recognizes congruent triangles (PFG and FKD) and then moves the area of triangle PFG to triangle FKD, which can be seen in **Figure 4** when using a stylus to point out where triangle PFG would go, which is where square FGQK would be formed.

Table 5. Organization of the analyses of D1's mathematical work

Subtasks	Episodes	Sequence of mathematical actions
ST 1. Configuration as a triangle	E1. Congruence of triangles	- Drag point S - Identify congruent triangles - Transfer areas
	E2. Posing a conclusion	- Show that the ratio of the areas is ¼
ST2. Configuration as a square	E3. Area calculation	- Generate a square by dragging point S to determine its area - Resolve on the relationship between areas
ST3. Configuration as a trapezoid	E4. Congruence of triangles	- Generate a trapezoid by dragging point S - Identify triangle congruence - Reconfigure the trapezoid area
	E5. Posing a conclusion	- Resolve on the relationship between areas
ST4. Conclusion	E6. Conclusion from particular cases	- Resolve on the relationships between areas in general

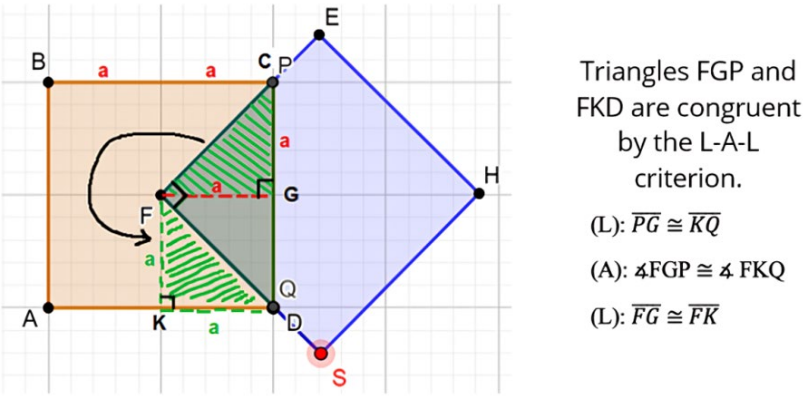


Figure 4. Subtask configuration as a triangle performed by D1 (image taken from a teacher's solution, used with informed consent as part of the research)

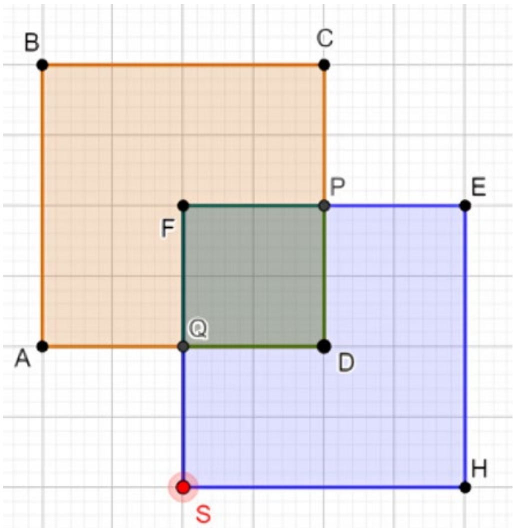
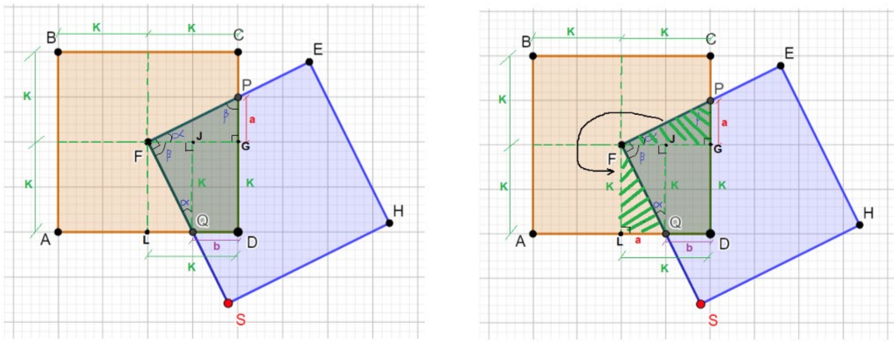


Figure 5. Subtask configuration as a square performed by D1 (image taken from a teacher's solution, used with informed consent as part of the research)

Then, in E2, D1 transfers the area of the triangle in PFG to determine that the area of square FGQK is a fourth of square ABCD.

ST2 is called configuration as a square. In E3, D1 calculates the area of the intersection region. To do this, point S is dragged until the intersection of squares ABCD and EFGH become a square (FPDQ). This can be observed in **Figure 5**. Then, by observing **Figure 5**, it is determined that the area of square FPDQ is a fourth of the area of square ABCD.



a b

Figure 6. Subtask configuration as a trapezoid performed by D1 (image taken from a teacher's solution, used with informed consent as part of the research)

In ST3, called configuration as a trapezoid, D1 drags point S in episode E4 until the intersection of squares ABCD and FPDQ form a trapezoid. Then, D1 identifies congruence of triangles as evidenced in part a in **Figure 6** where angles are added using the tablet's stylus to check the congruence between triangles FPG, FJQ and FLQ in order to then reconfigure the area of the trapezoid into the area of square FGDL, as evidenced in part b in **Figure 6**. Finally, in E5, D1 determines that the area of square FGDL is a fourth of square ABCD.

Based on the first three subtasks, D1 answers ST4, that is, D1 comes to a general conclusion, E6, in which D1 indicates that the area of the figure formed at the intersection of squares ABCD and FEHS is always a fourth of the area of square ABCD.

Analyzing the circulation, ST1 E1 shows evidence of a work that activates the instrumental genesis where D1, by means of the GeoGebra artifact, forms a triangle in the intersection of the squares, which also activates visualization, the iconic one in particular. The discursive genesis is activated because, from D1's referential on triangle congruences, it is proven that triangles PFG and FKD are congruent. Then, the semiotic genesis is activated, since D1 makes a figural treatment by dividing triangle PFG into two triangles and then translates triangle PFG to reconfigure the area of the larger triangle into a square, carrying out an iconic visualization. This indicates that the semiotic-discursive plane [Sem-Dis] was activated. In E2, the semiotic genesis is activated by means of visualization; D1 generalizes that the area of intersection is a fourth of square ABCD. This is also evidenced in the explanation of the procedure, when it says that

> "if we observe this triangle, it makes up a quarter of the area of this square (ABCD), which can be seen at first glance".

In ST2, the instrumental genesis is activated in E3; by means of the GeoGebra artifact, D1 forms a smaller square at the intersection of the squares. In addition, the semiotic genesis is activated by realizing (iconic visualization) that the area of intersection is a fourth of square ABC, as it is mentioned in the explanation of the procedure, "*when the intersection of these two areas is a smaller square, and we can also observe that this square (FPDQ) represents a fourth of the larger square (ABCD)*".

In ST3, the instrumental genesis of D1 is activated in E4 by means of the GeoGebra artifact, forming a trapezoid at the intersection of the squares; visualization is also activated, the iconic one in particular. In addition, the discursive genesis is activated because, from D1's referential on triangle congruences, it is proven that the triangles are congruent due to the L-A-L case, which is evidenced when D1 mentions that "*Therefore, it can be stated that triangles QJF and FGP are congruent due to the L-A-L congruence case*" in the explanation of the procedure. Then, the semiotic genesis is activated by reconfiguring the area of the trapezoid into triangles and rectangles; that is, a mereological visualization is evidenced (Duval, 2005). In E5, the semiotic genesis is activated, and by means of visualization, D1 generalizes that the area of the intersection is a fourth of square ABCD.

In ST4, the semiotic-discursive plane [Sem-Dis] is activated in episode E6 when, based on the different cases, D1 indicates that *for any shape of the figure formed at the intersection of the two squares, the area of the intersection is one fourth of the area of square ABCD*.

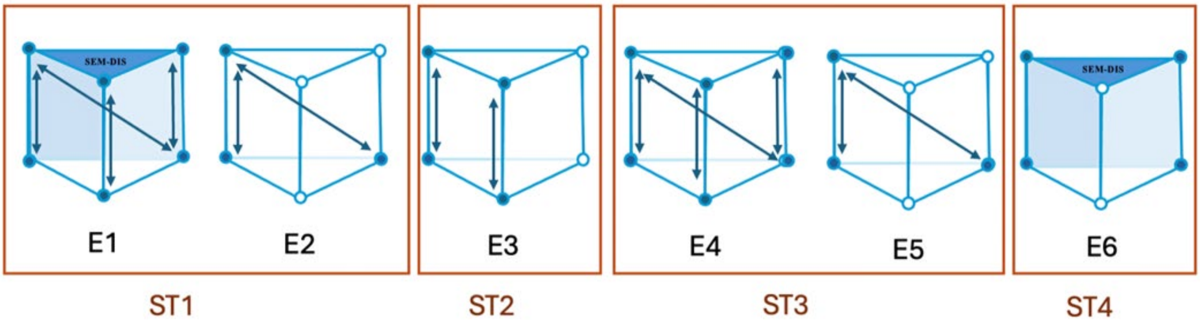


Figure 7. Global description of D1's personal MWS in each subtask (generated by the authors)

In the bottom-up analysis, there is a global description of the work using the MWS diagram (**Figure 7**).

In D1's mathematical work, a predominance of semiotic and discursive geneses is observed. In the overall analysis presented, the data show the relationship between the components of D1's personal MWS, which does not necessarily refer to a specific order or direction.

Case 2. D2's Personal Mathematical Working Space

In this task developed by D2, the emphasis is on iconic visualization and numerical calculations to validate the initial statement made.

In the top-down analysis, it is identified that the mathematical work of D2 is framed in an "initial conjecture" since the following is mentioned:

> The relationship is that the measurement of the area of the square is four times the area formed at the intersection, since F is the midpoint of one of the squares and A is the midpoint of another square, so the various configurations (triangle, square and trapezoid) are ¼ of the square.

Based on the above, the figurative representation of the conjecture made by D2 is shown in **Figure 8**.

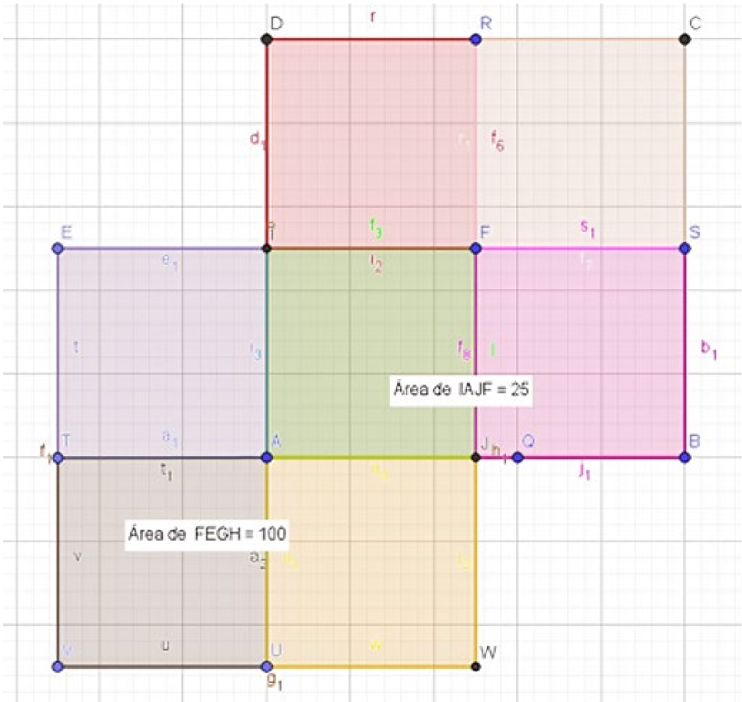


Figure 8. Figural representation of D2's conjecture (image taken from a teacher's solution, used with informed consent as part of the research)

Table 6. Organization of the analysis of D2's mathematical work

Subtasks	Episodes	Mathematical actions
ST 1. Configuration as a triangle	E1. Calculation of the area of the triangle	- Do an experimental test (GeoGebra dragging) - Calculate and justify (area of the square and triangle)
ST2. Configuration as a square	E2. Calculation of the area of the square	- Calculate by doing operations on the figure with GeoGebra
ST3. Configuration as a trapezoid	E3. Calculation of the area of the trapezoid	- Calculate in relation to other configurations
ST4. Conjecture validation	E4. Justification of the area of each configuration	- Justify with calculations and GeoGebra

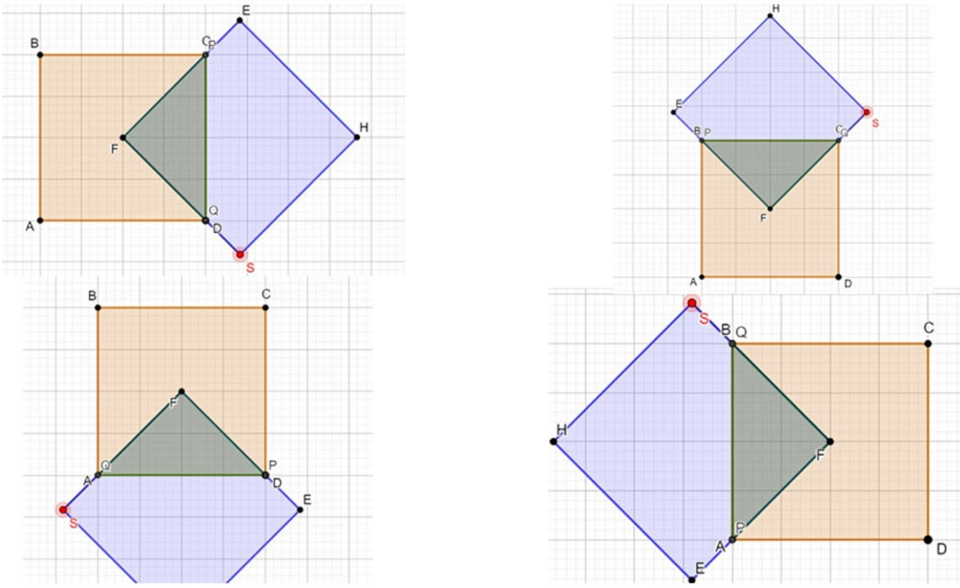


Figure 9. Triangle configurations (image taken from a teacher's solution, used with informed consent as part of the research)

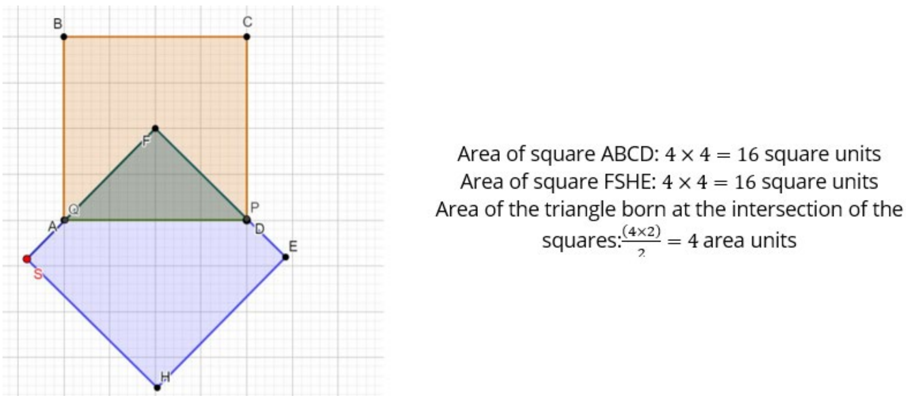


Figure 10. Calculation of the area of triangle QFP (image taken from a teacher's solution, used with informed consent as part of the research)

In each subtask (ST) it was identified that the mathematical work done by D2 had only one episode. **Table 6** shows the subtasks, episodes and mathematical actions performed by D2.

In ST1, called "configuration as a triangle", D2 uses the GeoGebra dragging function in E1 to move movable point S so that the intersection of squares ABCD and EFSH configures isosceles triangles, as shown in **Figure 9**.

Then, D2 performs an experimental test and perceptually verifies the initial conjecture for the isosceles triangle. Likewise, D2 calculates the area of squares ABCD and FSHE, and verifies that both have an area of 16 area units, as shown in **Figure 10**.

Next, D2 uses the formula for the area of a triangle to calculate its area and concludes that it is 4 area units. This shows that, in addition to perceptual validation, D2 uses the grid as a unit of measurement and the formulas of the area of the square and triangle, respectively.

In E2 of ST2, "configuration as a square", D2 uses GeoGebra's dragging function at point S and intercepts squares ABCD and EFSH to configure a square, which is shown in **Figure 11**.

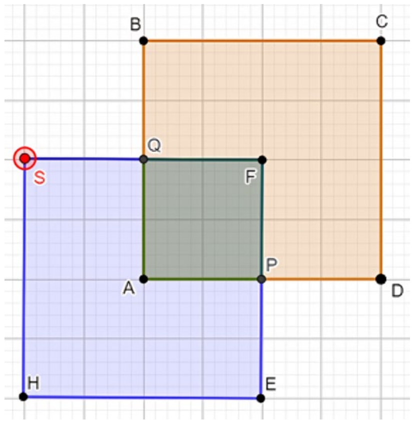


Figure 11. Configuration of a square (image taken from a teacher's solution, used with informed consent as part of the research)

Also, in this episode D2 performs the perceptual validation of the conjecture, uses the formula of the area of a square, and calculates the area of squares ABCD and FSHE. Based on **Figure 11**, the teacher uses the area formula and calculates the area of squares ABCD and FSHE, explaining the following:

> The area of square ABCD:$4 \times 4 = 16$ area units. The area of square FSHE: $4 \times 4 = 16$ area units. The area of the square formed by the intersection of the two squares: $2 \times 2 = 4$ area units.

These actions allow D2 to explain that squares ABCD and FSEH have equivalent areas, i.e., 16 square units, and that, consequently, the area of square AQFP (also using the area formula) formed by the intersection of the two squares is 4 area units.

In ST3, called "configuration as trapezoid" in E3, GeoGebra's dynamics is used to drag point S to the position shown in **Figure 12** in order to generate asymmetric trapezoid AQFP.

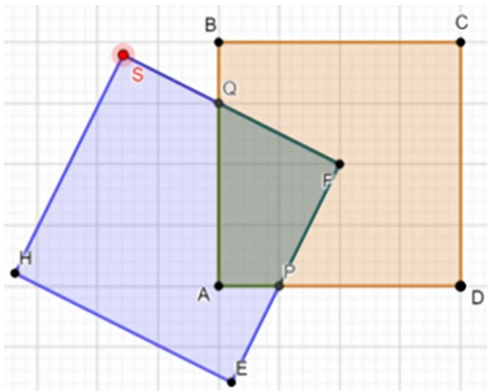
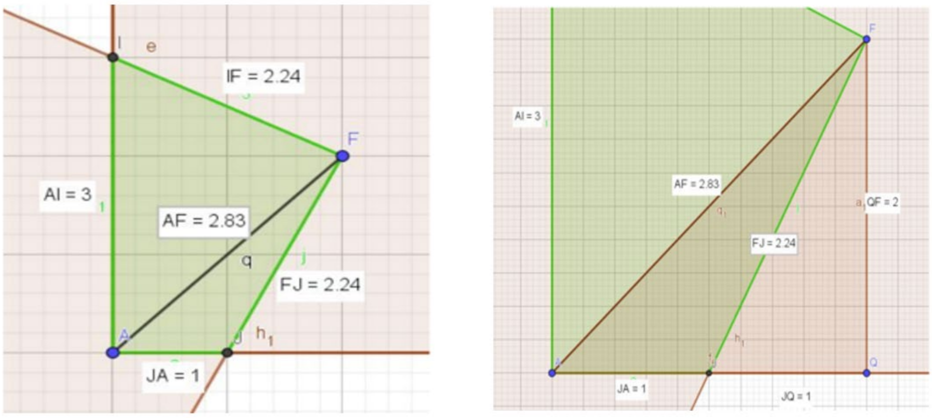


Figure 12. Trapezoid area (image taken from a teacher's solution, used with informed consent as part of the research)

Subsequently, D2 draws an auxiliary line with GeoGebra's segment tool to create segment AF on trapezoid IFJA, and with the use of the distance tool measures segments AI, IF, AF, FJ and JA, as shown in part a in **Figure 13 (a)**. Then, D2 draws auxiliary lines to create right triangle AFQ, as shown in part b in **Figure 13 (b)**, and measures segment FQ. D2 points out that the area of triangle AFJ is half the area of triangle AFQ.



a b

Figure 13. AQFP trapezoid area (image taken from a teacher's solution, used with informed consent as part of the research)

In E3, D2 shows that the area of triangle AFJ is half the area of triangle AFQ. This statement could be shown with GeoGebra tools or with the formula to determine the area of the right triangle.

Based on the previous subtasks, in ST4, named "conjecture validation" in E4, D2 confirms the initial conjecture about the relation of areas in the different configurations and explains that the interaction with GeoGebra allows D2 to show the statement, to observe that if the intersection (cases: square and triangle) is quadrupled around the center of square ABCD, the initial square is formed. In the case of the asymmetric trapezoid, the same thing happens as in the previous cases because the area of the asymmetric trapezoid is 4 area units.

Analyzing the circulation, in the "initial conjecture" episode, D2 poses the conjecture of the area relation of the intersection by dragging point S using the GeoGebra digital artifact. In ST1, which includes E1, D2's mathematical actions show the activation of the referential in relation to the area of a square and triangle to calculate those areas in order to validate the initial conjecture (triangle case). D2 also activates the semiotic genesis, which is evident when D2 makes the perceptual validation and uses the grid as units of measurement; consequently, the semiotic-discursive [Sem-Dis] plane was activated.

In ST2, the semiotic genesis is activated because D2 performs the perceptual validation of the conjecture, uses the formula of the area of a square as a symbolic artifact, and calculates the area of squares ABCD and FSHE. In addition, the teacher mobilizes the referential to calculate the area of the respective configured square.

Regarding ST3, the instrumental genesis is activated in E3 because D2 uses GeoGebra to drag point S to the desired position in order to generate trapezoid AQFP. Subsequently, D2 activates the semiotic and instrumental genesis, i.e., the [Sem-Ins] plane, because D2 divides the figure into subfigures, using different GeoGebra tools to generate segment AF, right triangle AFQ, and AFJ. D2 also uses the measurement tool to measure the area of the respective triangles. In addition, D2 activates the theoretical referential by stating that the area of triangle AFJ is half the area of triangle AFQ.

Regarding ST4, the initial conjecture is validated in E4, since the discursive and semiotic geneses are activated in D2 because, after having performed mathematical actions to prove the relationship between areas, it is shown that the area of the square is four times the area formed in the intersection, as isosceles triangle, square and trapezoid, which means that the area of those figures is ¼ of the area of the square. In the bottom-up analysis, the global description of the mathematical work is performed by means of the MWS diagram (**Figure 14**).

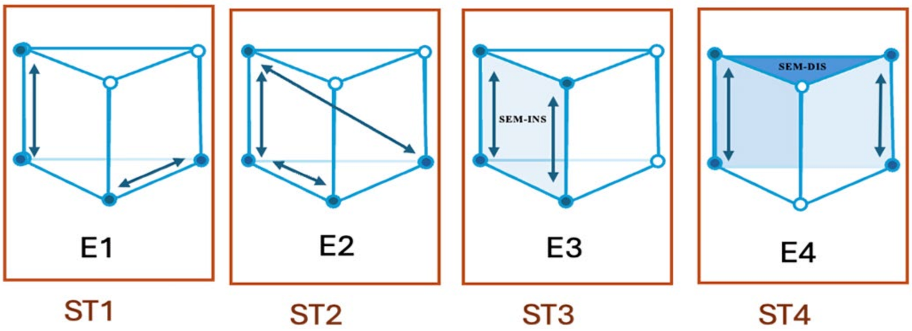


Figure 14. Overall description of D2's personal MWS in each subtask (generated by the authors)

DISCUSSION

Table 7 shows a summary of D1's personal MWS, including subtasks, episodes and elements of the MWS that were mobilized to solve the proposed task.

Table 7. Summary of D1's personal MWS

Subtasks and episodes (E)		Types of geneses, components, and activated vertical planes						
		Semiotic genesis		Instrumental genesis		Discursive genesis		Vertical plane
		Representation	Visualization	Artifact	Construction	Referential	Proof	
ST1	E1	X	X	X	X	X	X	[SEM-DIS]
	E2	X	X			X		-
ST2	E3	X	X	X	X			-
ST3	E4	X	X	X	X	X	X	-
	E5	X	X			X		-
ST4	E6	X	X			X	X	[SEM-DIS]

Based on the results obtained from the mathematical works of the teachers (D1 and D2), it is evident that the personal MWS of each one is different. This is observed from the moment in which D1studies each case (intersection as a triangle, square and trapezoid) to then draw a conclusion on the relationship of the areas. On the other hand, D2 starts from a conjecture that arises from manipulation, specifically from the dragging point S to then validate it based on the cases (intersection as a triangle, square and trapezoid).

When analyzing D1's personal MWS, one aspect to highlight is the predominance of discursive-graphical proof in the sense of Richard (2004), which has been reported in recent studies (e.g., Henríquez-Rivas & Verdugo-Hernández, 2023). That is, it shows a reasoning that is organized based on discursive propositions and graphic representations with the aim to prove. In turn, it is also observed that there is a predominance of semiotic genesis, where visualization is iconic in the sense of Duval (2005), associated with the perceptual recognition of figures. In episodes E1 and E4, D1 uses triangle congruence to transfer areas, which indicates the use of pragmatic evidence in the sense of Balacheff (1987), given that the justifications are more linked to the representations than to the arguments.

Table 8 shows a summary of D2's personal MWS, which presents the subtasks with their respective episodes and MWS elements that were mobilized to solve the proposed task.

Table 8. Summary of D2's personal MWS

Subtasks and episodes (E)		Types of geneses, components, and activated vertical planes						
		Semiotic genesis		Instrumental genesis		Discursive genesis		Vertical plane
		Representation	Visualization	Artifact	Construction	Referential	Proof	
ST1	E1	X	X	X		X		-
ST2	E2	X	X	X		X		-
ST3	E3	X	X	X	X			[SEM-INS]
ST4	E4	X	X			X	X	[SEM-DIS]

In the analysis of D2's personal MWS, it stands out the fact that it is a work that emphasizes iconic visualization (Duval, 2005), as well as the numerical calculations to validate the initial statement made. Like D1, D2 also performs pragmatic tests using representations, although there is also evidence of the use of intellectual proofs, as when D2 proves that two triangles are congruent. In the personal MWS of both teachers (D1 and D2), the semiotic and discursive geneses are prevalent. Both show a predominance of iconic visualization processes (Duval, 2005) and a pragmatic test (Balacheff, 1987).

As for their reasoning, D1's was more inductive when analyzing particular cases and then finished. In addition to the pragmatic tests, D1 also used intellectual tests to identify that two triangles were congruent, so it could be said that D1 is in geometric paradigm G2. However, D2 had a more deductive reasoning, since first a conjecture was made and then validated it through specific cases. Although these procedures were more related to calculation and formula application, it could be said that D2 is in geometric paradigm G1. From the above, it can be determined that the mathematical work of both teachers is in different geometric paradigms.

It is worth mentioning that D2 and D1 both used GeoGebra, but on different electronic devices; D2 used a laptop while D1 a tablet. This allowed D1 to combine dynamic representations of the software with representations made with the pencil tool, which allowed D1 to draw lines that helped check the congruence of triangles to be able to translate the figures. The latter could be the reason why the mathematical work of D2 shows the activation of the semiotic and instrumental geneses. It is worth noting that the GeoGebra digital artifact was used differently by both. For example, in some episodes D1 used GeoGebra's pencil tool as a traditional pencil, allowing procedures (algebraic in this case), while D2 used tools such as area measurement, segments for auxiliary lines, etc. However, for both of them it was fundamental to relate the different configurations of the figures requested in the task.

CONCLUSIONS

The research shows that the personal MWS is different in each subject of the study. Despite the fact that they worked on the same task, different episodes were generated, activating different geneses and vertical planes, that is, different elements of the MWS.

Although both participants used GeoGebra, it is evident that they do different mathematical works, which indicates that the use of different technological devices (artifacts), such as a tablet or a computer, could modify the MWS of the subjects. This is because, on the one hand, while working with the tablet, D1 was able to draw auxiliary lines to the figure as if it were a virtual notebook, while the use of the GeoGebra application on the computer facilitated D2 to use the dragging function, which benefited the development of iconic visualization. In addition, each device has different historical intelligence because they are built based on an epistemology, which is the result of a process of human intellectual construction, and epistemological validity that allows the representation of mathematical concepts depending on the potential or limitations of the artifact used. Consequently, we noted that the semiotic and instrumental geneses are activated at different moments of D1 and D2's mathematical work.

Regarding the limitations of this research, one of these is related to the data collection and analysis from the mathematical work of two teachers, so in the future a larger number of cases could be analyzed or use a mixed research approach in order to consider qualitative and quantitative data and, eventually, include these results. On the other hand, there are limitations associated with the single-case design, and knowing if similar results are obtained in multiple cases in different contexts. We propose to replicate the methodological design of this research and the input developed in future research.

Finally, future research is encouraged to focus on personal MWS in the domain of geometry, considering other mathematical objects and other artifacts that allow the identification of personal MWS by using different technological artifacts. Also, it is recommended to consider different dimensions of the figure (squares of equal side units); this could perhaps allow the actions of the subjects not to focus on visualization but on proof justification. A recent literature review study focused on the MWS theory (Panqueban et al., 2024) could provide guidance in this type of study.

Author contributions: JVFS: conceptualization, methodology, investigation, data curation, writing – original draft; **DJG-C:** conceptualization, methodology, investigation, data analysis, writing – review & editing; **CH-R:** conceptualization, methodology, validation, writing – review & editing; **AK:** conceptualization, supervision, writing – review & editing. All authors approved the final version of the article.

Funding: Funding: This study was funded by the Vicerrectorado de Investigación (VRI) at the Pontificia Universidad Católica del Perú through grant DFI-2022-0890.

Acknowledgments: The authors thank the Instituto de Investigación sobre la Enseñanza de las Matemáticas (IREM-PUCP) and the Red Iberoamericana de Investigación en Trabajo Matemático (RIITMA) at the Pontificia Universidad Católica del Perú for their institutional support. C. Henríquez-Rivas also acknowledges the Agencia Nacional de Investigación y Desarrollo (ANID), Chile, for funding through Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) de Iniciación 2023, No. 11230523.

Ethics declaration: This study was approved by the Research Ethics Committee for Social Sciences, Humanities and Arts at the Pontificia Universidad Católica del Perú on 16 August 2022 with approval code 080-2022-CEI-CCSSHHyAA/PUCP. Participants voluntarily agreed to participate in the study and signed an informed consent form prior to the data collection process. The consent form explained the objectives of the research, the voluntary nature of participation, and the use of the collected data exclusively for research purposes. To guarantee confidentiality, participants were anonymized using codes (D1 and D2). No personal identifying information was included in the dataset or in the publication. All data were stored securely and accessed only by the research team.

AI statement: The authors declare that no generative artificial intelligence tools were used in the design, analysis, or interpretation of the research data. AI-based tools were not used to generate the scientific content of this manuscript.

Declaration of interest: The authors declared no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

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